

◇ Useful formulas $\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}}$

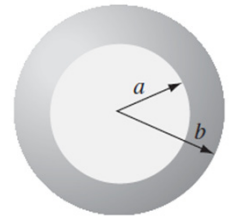
$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

$$\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}$$

1. (a) Find the divergence of the function $\mathbf{v} = (r \cos \theta) \hat{\mathbf{r}} + (r \sin \theta) \hat{\boldsymbol{\theta}} + (r \sin \theta \cos \phi) \hat{\boldsymbol{\phi}}$. (7%)
 (b) Test the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R , resting on the xy plane and center at the origin. (7%)
 (c) Find the curl of $\mathbf{v}$. (6%)

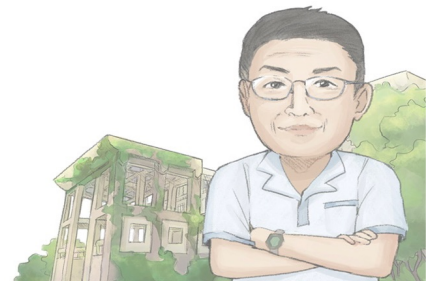
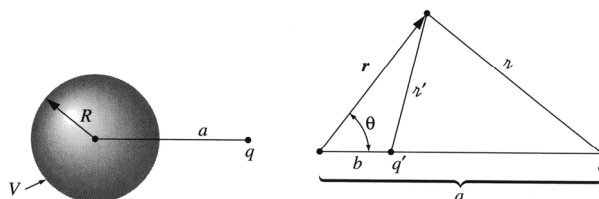
2. Consider a thick spherical shell with the charge density:

$$\rho = \begin{cases} \frac{\rho_0 r}{a}, & r < a \\ \frac{\rho_0 a^2}{r^2}, & a < r < b \end{cases}$$

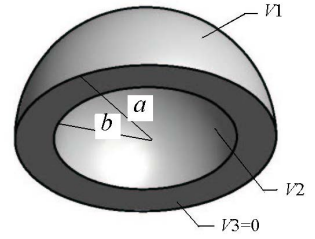


and a perfect conducting shell with surface charge density σ_b is placed at $r = b$, making $\mathbf{E} = 0$ for $r > b$. Find the electric field in the three regions:

- (a) $r < a$, (5%)
 - (b) $a < r < b$, (5%)
 - (c) $r > b$, (5%)
 - (d) Find the value of σ_b . (5%)
3. A point charge q is situated at distance a from the center of a conducting sphere of radius R . The sphere is maintained at the constant potential V .
 (a) If $V=0$, find the position and value of the image charge. (5%)
 (b) If $V=V_0$, find the potential outside the sphere. (5%)
 (c) Find the electric field on the surface of the metal sphere. (5%)
 (d) Find the surface charge density and the total charge on the metal sphere. (5%)
 [Hint: 1. use the notations shown below. 2. Assume q lays on the z -axis]



4. Suppose the potential on the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a, \theta) = V_0(5 \cos^3 \theta - 3 \cos \theta)$, $V_2(b, \theta) = 0$, $V_3(r, \pi/2) = 0$. V_0 is a constant.



- Show the general solution in the region $b \leq r \leq a$. (4%)
- Determine the potential in the region $b \leq r \leq a$, using the boundary conditions. (10%)
- Calculate the electric field on the surface of the outer shell $\mathbf{E}(r = a)$. (6%)

[Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, and $P_3(x) = (5x^3 - 3x)/2$.]

5. An idea electric dipole $\mathbf{p}$ is situated at the origin, and points in the z direction. An electric charge q , of mass m , is released from rest at a point in the xy plane. The potential of the dipole is $V(\mathbf{r}) = (1/4\pi\epsilon_0)(p \cos \theta / r^2)$ and the gravitational force points in the $-z$ direction.
- Find the electric force between the dipole and the charge. (8%)
 - Find the total force (electrical and gravitational) on the charge. (6%)
 - Find the total potential energy. (6%)

